Compound Damped Pendulum: An Example

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Abstract: In this article, we use an energy approach applied to damped pendulum oscillators enjoying three types of damping: quadratic, viscous, and a combination of these two called compound. The energy approach applies to many types of oscillators, the pendulum was chosen for its nonlinearity. We calculate the maximum amplitudes of oscillations without solving the model differential equation and compare the dissipation of energy by obtaining a relationship between the sum of kinetic and potential energies and displacement alone, bypassing velocity. The compound model is compared to a viscous damped model and a quadratic damped model.

Key Words: Quadratic damping, viscous damping, energy and dissipation, damped pendulum

1 Introduction.

The author has begun an investigation of damping in (strictly dissipative) oscillator equations using an energy approach [1], [2], [3]. This approach is suitable for classroom discussion and computer laboratory explorations. What makes this investigation interesting is that the damping term for dry friction or for quadratic damping in the oscillator equations involves a jump discontinuity whenever the velocity is zero, since the sign of the damping must change in order to always oppose the direction of motion. This is even taken into account when the damping is viscous.

In this article, we use a numerical approach to investigate damping of the pendulum. The common viscous damping assumption for the harmonic oscillator is more of a mathematical convenience as it leads to a linear model having a closed form solution. The author knows of no such closed form solution for the viscous damped pendulum, and thus there is no mathematical reason to prefer viscous damping over the perhaps more realistic quadratic damping. However, there does appear to be an alternative to either viscous or quadratic damping, and that is a combination of the two, which we call compound damping. The idea is that large deflections from equilibrium experience quadratic damping while small deflections experience viscous damping. Perhaps the main point here is that mathematically and numerically, it is no more expensive to investigate one

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form of damping than the other and compound damping arguably gives rise to a more realistic model.

This approach facilitates finding the energy of an initial value problem as a function of displacement and this permits the building of a graphic for the visual comparison of the types of damping and the corresponding dissipation. Moreover, this quantifies dissipation of energy and thus allows for comparison of initial value problem solutions and models.

In particular, the viscous damped pendulum model
\[ \ddot{x} + \delta \dot{x} + \sin x = 0, \tag{1} \]
and the quadratic damped pendulum model
\[ \ddot{x} \pm \varepsilon \dot{x}^2 + \sin x = 0. \tag{2} \]
are compared with a compound damped pendulum model
\[ \ddot{x} + \delta \dot{x} \pm \varepsilon \dot{x}^2 + \sin x = 0. \tag{3} \]
The \pm indicates that the damping force always opposes the direction of motion and thus has to change sign when the sign of the velocity changes (this is not necessary in the viscous damped model).

\subsection*{1.1 Friction}

In this article, we only consider strictly dissipative frictional forces that oppose the direction of the motion. Consequently a frictional force always has the opposite sign of the velocity. To interpret such a force with the correct sign, the equation of motion becomes
\[ \ddot{x} + \delta \text{Sgn}(\dot{x})q(\dot{x}) + \sin x = 0. \tag{4} \]
(the mass of the system has been normalized to be 1). The term \( \text{Sgn}(\dot{x})q(\dot{x}) \) is called the damping function, the positive constant \( \delta \) is material or medium dependent; the signum function \( \text{Sgn}(\dot{x}) \) is defined by
\[ \text{Sgn}(\dot{x}) = \frac{\dot{x}}{|\dot{x}|}, \tag{5} \]
where it is understood that \( \text{Sgn}(0) = 0 \). We shall simplify the notation by writing \( \delta \text{Sgn}(\dot{x}) = \pm \delta \) where it also is understood that the plus sign is taken when the velocity is positive and the minus sign is taken when the velocity is negative.

The equations for a dissipative oscillator, using Newton’s Law with mass normalized to be 1, with viscous, quadratic, and compound damping are respectively
\[ \ddot{x} + \pm \delta |\dot{x}| + \sin x = 0, \]
\[ \ddot{x} + (\pm \delta) |\dot{x}|^2 + \sin x = 0, \tag{6} \]
\[ \ddot{x} + \pm \delta |\dot{x}| + \pm \varepsilon |\dot{x}|^2 + \sin x = 0. \]
Note the use of absolute value as viscous damping is proportional to speed and quadratic damping is proportional to the square of speed. These equations become (1), (2), and (3) above.

2 Energy

We begin with the damped pendulum model

\[ \ddot{x} + \pm \delta q(\dot{x}) + \sin x = 0, \]  

(7)

Multiply (7) by \( \dot{x} d\tau \) and integrate from \( \tau = 0 \) to \( \tau = t \), to obtain

\[ \int_0^t \ddot{x} \dot{x} d\tau + \int_0^t \pm \delta q(\dot{x}) \dot{x} d\tau + \int_0^t \sin x \dot{x} d\tau = 0. \]  

(8)

We let \( y = \dot{x} \). Integrating:

\[ \frac{y^2}{2} - \frac{y_0^2}{2} + \cos x_0 - \cos x = - \int_0^t \pm \delta q(\dot{x}) \dot{x} d\tau \]  

(9)

where of course \( x_0 = x(0) \) and \( y_0 = y(0) \). Equations (8) and (9) are each two equations depending upon the sign of the velocity \( \dot{x} \). Note that there is a jump discontinuity when \( \dot{x} = 0 \). This means that the motion is divided into intervals bounded by the conditions that the velocity is zero and between endpoints is of constant sign.

The classical energy approach is to set

\[ E(x, y) = \frac{y^2}{2} - \cos x \]  

(10)

On the right-hand-side of the equation, we see the sum of terms representing the kinetic energy and potential energy. So equation (10) can be rewritten, for initial conditions \((x_0, y_0)\),

\[ E(x, y) = \frac{y_0^2}{2} - \cos x_0 - \int_0^t \pm \delta q(\dot{x}) d\tau d\tau. \]  

(11)

Our goal is to have an expression for \( E(x, y) \) that solely depends upon \( x \) and that does not involve the dissipation integral integral on the right-hand-side of equation (11). In general this dissipation integral can not be easily evaluated.

Writing equation (7) as a system

\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= \mp \delta q(y) - \sin x
\end{align*} \]  

(12)

and recognizing that

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \]  

(13)
we have to the first order initial value velocity equation

\[ y \frac{dy}{dx} + \pm \delta q(y) = -\sin x, \quad y(x_0) = y_0. \]  \hspace{1cm} (14)

This velocity equation solution is a function of displacement and thus we have the energy equation (10) as a function of displacement alone. Recall however, the motion is divided into displacement intervals bounded by the conditions that the velocity is zero at the endpoints and between endpoints is of constant sign.

3 Compound damped pendulum

We begin with the compound damped pendulum model

\[ \dot{x} + \delta y + \pm \varepsilon y^2 + \sin x = 0, \]  \hspace{1cm} (15)

where \( \delta = 1/4, \varepsilon = 1/2 \) and \((x_0, y_0) = (1, 1)\), these parameter values being considered representative. In order to display the energy function (10) for this example we need to determine the closed displacement intervals at whose endpoints the velocity is zero and between is of constant sign. These intervals correspond to intervals over which the solution is rising from a local minimum to a local maximum or falling from a local maximum to a local minimum. In Figure 1 we show plots of the displacement \( x(t) \) and velocity \( y(t) \) for \( 0 \leq t \leq 50 \) obtained by solving the initial value problem (15).

In Figure 2 we show the full trajectory for the initial value problem in the phase plane, indicating the initial point, and highlighting the first ten crossing points, \((x_i, 0)\), where the trajectory crosses the \( x \)-axis. These \( x_i \) values delineate the
displacement intervals.

Figure 2. The trajectory, initial point and crossing points.

The crossing points are obtained in an interactive manner using Mathematica [4] although most computer algebra systems would suffice. For our initial value problem, the first displacement interval is \([x_0, x_1]\); over this interval \(y \geq 0\), and \(x_1\) is the abscissa of the first point on the \(x\)-axis reached from the initial point \((1, 1)\) traveling clockwise on the trajectory. We find this point and calculate a numerical solution \(y_1(x)\) simultaneously by solving the initial value problem

\[
y_1 \frac{dy_1}{dx} + \frac{1}{4} y_1 + \frac{1}{2} y_1^2 = -\sin x, \quad y_1(x_0) = y_0,
\]

for \(x_0 = 1 \leq x \leq 3\). Mathematica does not like solving this equation over \([1, 3]\) since \(y_1 = 0\) is a singular point for equation (16), and indeed Mathematica returns the value \(x_1 = 1.38050\) as the farthest it can solve the equation before encountering the singularity.

The next interval is \([x_2, x_1]\) over which \(y \leq 0\). The abscissa \(x_2\) of this next crossing point is found by following the trajectory from \((x_1, 0)\) below the \(x\)-axis moving clockwise curling up to the point \((x_2, 0)\). This point is found by solving the initial value problem

\[
y_2 \frac{dy_2}{dx} + \frac{1}{4} y_2 - \frac{1}{2} y_2^2 = -\sin x, \quad y_2(x_1) = -0.0001,
\]
over the interval \([-2, x_1]\). Note the change in sign on the squared term to account for \(y \leq 0\) over the interval and the initial condition. This initial value is chosen to avoid setting \(y_2(x_1) = 0\) and having *Mathematica* refuse to solve due to a singularity when \(y_2 = 0\). *Mathematica* returns \(x_2 = -0.47366\) as the value at which \(y_2\) becomes zero.

The next initial value problem to solve is

\[
y_3 \frac{dy_3}{dx} + \frac{1}{4} y_3 + \frac{1}{2} y_3^2 = -\sin x, \quad y_3(x_2) = 0.0001
\]  

(18)

over the interval \([x_2, 2]\) to determine \(x_3\). Again note the change in sign and the initial condition.

Proceeding in this iterative way, we produce the desired crossing values and determine the displacement intervals. Over each of these intervals we have as associated function \(y_i(x)\) and thus an energy function

\[
E_i(x) = \frac{y_i^2(x)}{2} - \cos x.
\]  

(19)

The first ten crossing points are listed in Table 1.

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(x_5)</th>
<th>(x_6)</th>
<th>(x_7)</th>
<th>(x_8)</th>
<th>(x_9)</th>
<th>(x_{10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.38050</td>
<td>-0.47366</td>
<td>0.24817</td>
<td>-0.14595</td>
<td>0.09064</td>
<td>-0.05801</td>
<td>0.03780</td>
<td>-0.02491</td>
<td>0.01653</td>
<td>-0.01102</td>
</tr>
</tbody>
</table>

Table 1

Piecing together plots of the energy functions produces a visual graphic of
energy dissipation. In Figure 3, we show the energy dissipation for our example.

![Compound damping energy displacement](image)

Figure 3. Energy displacement plot for the compound damped pendulum example.

It is of interest also to tabulate the drops of energy from one interval to the next. In Table 2 we list the energies at each of the crossing points.

| $x_1$ | $E_1$ = -0.18915 |
| $x_2$ | $E_2$ = -0.889905 |
| $x_3$ | $E_3$ = -0.969364 |
| $x_4$ | $E_4$ = -0.998318 |
| $x_5$ | $E_5$ = -0.999296 |
| $x_6$ | $E_6$ = -0.99969 |
| $x_7$ | $E_7$ = -0.999863 |
| $x_8$ | $E_8$ = -0.999939 |

Table 2

After about three oscillations, the energy has become approximately $-0.999$, which is very close to being the minimum level for this example.

### 3.1 Potential energy

A potential energy associated with a conservative restoring force, that is, a force depending solely upon displacement, $f(x)$, is a function $V(x)$ that satisfies

$$\frac{dV}{dx} = -f(x).$$  \hspace{1cm} (20)

Thus

$$V(x) = -\int_{x_0}^{x} f(\xi)d\xi + V(x_0)$$ \hspace{1cm} (21)
where \( V(x_0) \) is an arbitrary constant of integration. Usually this constant of integration is chosen to set the zero of the potential energy. In the above example, we could have added the constant 1 so that the minimum energy for the example would have been 0 and not \(-1\), resulting in merely a vertical offset shift of the plot.

## 4 Viscous damping

Consider now the viscous damped equation

\[
\ddot{x} + \frac{1}{4} y + \sin x = 0. \tag{22}
\]

with initial conditions \((x_0, y_0) = (1, 1)\). A plot of the solution \(x(t)\) and the velocity \(y(t)\) is shown in Figure 4.

![Viscous Damped Solution and Velocity](image)

Figure 4. The displacement and velocity for the viscous damped pendulum example.
The trajectory is shown with crossing points in Figure 5.

![Viscous Damped Trajectory](image)

Figure 5. The trajectory for the viscous damped example.

The associated velocity equation is

$$\frac{dy}{dx} + \frac{1}{4} y = -\sin x$$

Proceeding in exactly the same as for the compound example, we determine the new crossing points $x_i$ and solution intervals over which the velocity is of constant sign. This leads to computing energies over the intervals and producing
the energy dissipation plot shown in Figure 6.

![Figure 6. The energy dissipation plot for the viscous damped example.](image)

In Table 3 we list the energies at each of the new crossing points.

<table>
<thead>
<tr>
<th>Crossing Point</th>
<th>Energy Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1(x_1)$</td>
<td>-0.115677</td>
</tr>
<tr>
<td>$E_2(x_2)$</td>
<td>-0.614645</td>
</tr>
<tr>
<td>$E_3(x_3)$</td>
<td>-0.828076</td>
</tr>
<tr>
<td>$E_4(x_4)$</td>
<td>-0.922615</td>
</tr>
<tr>
<td>$E_5(x_5)$</td>
<td>-0.96504</td>
</tr>
<tr>
<td>$E_6(x_6)$</td>
<td>-0.98418</td>
</tr>
<tr>
<td>$E(x_7)$</td>
<td>0.992836</td>
</tr>
<tr>
<td>$E_8(x_8)$</td>
<td>0.996755</td>
</tr>
<tr>
<td>$E_9(x_9)$</td>
<td>0.99853</td>
</tr>
<tr>
<td>$E_{10}(x_{10})$</td>
<td>0.999334</td>
</tr>
</tbody>
</table>

Table 3

After approximately four and one half oscillations the energy value is about $-0.999$, close to the minimum energy level.

## 5 Quadratic damping

Now consider the quadratic damped pendulum equation,

$$\dot{x} + \frac{1}{2}y^2 + \sin x = 0.$$  \tag{24}
with \((x_0, y_0) = (1, 1)\) and \(\varepsilon = 1/2\). We plot the solution \(x(t)\) and the velocity \(y(t)\) in Figure 7.

We plot the solution \(x(t)\) and the velocity \(y(t)\) in Figure 7.

Figure 7. The solution and velocity for the quadratic damped example.

The trajectory is plotted in Figure 8 with the first sixteen new crossing points indicated, starting from the initial point \((1, 1)\) and proceeding clockwise.

The associated velocity equation is

\[
y \frac{dy}{dx} + \frac{1}{2} y^2 = -\sin x.
\]  

Again, proceeding in exactly the same as for the compound example, we determine the new crossing points \(x_i\) and solution intervals over which the velocity
is of constant sign. This leads to computing energies over these intervals and producing the energy dissipation plot shown in Figure 9.

![Figure 9. The energy dissipation plot for the quadratic damped example.](image1)

From this figure we see that after sixteen oscillations, the quadratic damped pendulum is still losing energy but very slowly. Of course this can be imagined simply by observing the plot of the numerical solution, but comparing plots of solutions often doesn’t permit an effective comparison of energy dissipation between examples. From Figure 10 however, we can see that the energy dissipation is very slow and visible oscillations are still apparent as far along as $t = 200$.

![Figure 10. Quadratic damped pendulum solution to the initial value problem for $0 \leq t \leq 200$.](image2)
In Table 4 we list the energies at each of the crossing points.

\[
\begin{align*}
E_1(x_1) &= -0.142015 & E_2(x_2) &= -0.771939 \\
E_3(x_3) &= -0.893107 & E_4(x_4) &= -0.93789 \\
E_5(x_5) &= -0.959386 & E_6(x_6) &= -0.971364 \\
E_7(x_7) &= -0.978722 & E_8(x_8) &= -0.983567 \\
E_9(x_9) &= -0.986926 & E_{10}(x_{10}) &= -0.989351 \\
E_{11}(x_{11}) &= -0.991158 & E_{12}(x_{12}) &= -0.99254 \\
E_{13}(x_{13}) &= -0.993622 & E_{14}(x_{14}) &= -0.994485 \\
E_{15}(x_{15}) &= -0.995184 & E_{16}(x_{16}) &= -0.995758
\end{align*}
\]

Table 4

After approximately five and one half oscillations the energy value is only about \(-0.99\), not so close to the minimum energy level as in the previous models. The energy change from one interval to the next is taking place in the fourth decimal place after seven oscillations.

6 Conclusions

The iterative procedure employing *Mathematica*’s ode solving routine, is very robust and permits building an energy dissipation portrait for an oscillator (not just the pendulum) by creating energy functions that depend solely on displacement. The iterative procedures introduced in [1] and [2] are different and slightly more tedious to apply. This numerical technique speeds the process up considerably as one computes the crossing points and the velocity and hence energy functions virtually simultaneously.

We like to point out that for the compound, viscous and quadratic damped pendulum equations discussed above, there are no known analytic solutions, yet through this investigation, we can determine the maximum deflections from equilibrium which certainly are the most significant features of an oscillator solution. Moreover, in studying any initial value problem associated with a damped oscillator equation, a plot of the energy dissipation yields quantitative information for comparison purposes that is otherwise only available through more tedious calculations.

References


1944 words,
10 figures
4 tables